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We will begin by examining the situation in which we have the ball very close to the opponent's goal line, but the clock is not a factor, so that we care only about the probability of scoring.

Suppose there is a continuum of possible defenses, ranging from exclusively focused on the run to exclusively focused on the pass. We consider only defenses on the "efficient frontier," in other words, defenses that are as effective as possible against the pass, subject to a given effectiveness against the run. We will index these defenses by the probability that a running play succeeds against it. Formally, let θ denote the defense against which a running play has probability θ of success. Let $f(\theta)$ denote the probability that a pass play succeeds against defense θ.

The defense can concentrate more on the run at the cost of lowered effectiveness against the pass, ore vice versa, but we assume this tradeoff is subject to diminishing returns: Formally, we assume that f is a convex function.

As explained in the article
`http://www.footballcommentary.com/nashequil.htm`,
the equilibrium defense θ^* satisfies $f(\theta^*) = \theta^*$. In other words, in equilibrium, run and pass plays must be equally likely to succeed.

The offense will use a randomized strategy in which it runs with probability p^* and passes with probability $1 - p^*$. We can determine p^* by the condition that θ^* is optimal against it: The solution to

$$\min_{\theta} \ p^* \theta + (1 - p^*) f(\theta)$$

must be $\theta = \theta^*$. This implies

$$p^* + (1 - p^*) f'(\theta^*) = 0$$

so that

$$p^* = \frac{-f'(\theta^*)}{1 - f'(\theta^*)}. \qquad (1)$$

In equilibrium, then, the defenders use defense θ^* and the offense uses a randomized strategy in which it runs with probability p^*.

The negative number $f'(\theta^*)$ represents the change in effectiveness against the pass per unit change in effectiveness against the run. If this is small in absolute value, then p^* is small, and in equilibrium the offense will usually pass. Conversely, if $f'(\theta^*)$ is large in absolute value, then p^* is large, and in equilibrium the offense will usually run.

Now suppose that it's near the end of the game, and we need a touchdown to win. We have first and goal very close to the opponent's goal line. Further suppose that there is more time remaining than we need, so that (other things being equal) we would prefer to run time off the clock. However, we assume that this preference is not so great that we choose to waste a down. We will show how to find the equilibrium strategies $(\hat{\theta}, \hat{p})$, and compare them to the strategies (θ^*, p^*) characterized earlier for the case in which the clock is not an issue.

Let w be the probability that we win the game if we score on first down. Let q_r denote the probability that we win the game if we run on first down but fail to score. Let q_p denote the probability that we win the game if we pass on first down but the pass is incomplete. Our assumptions imply that $w > q_r > q_p$.

Against defense θ, our probability of winning the game if we choose to run on first down is

$$\theta w + (1 - \theta) q_r,$$

whereas our probability of winning if we choose to pass is

$$f(\theta) w + (1 - f(\theta)) q_p.$$

As we explained in the article, the equilibrium defense will be the defense $\hat{\theta}$ against which running and passing give equal probabilities of winning the game. Hence

$$\hat{\theta} w + (1 - \hat{\theta}) q_r = f(\hat{\theta}) w + (1 - f(\hat{\theta})) q_p, \tag{2}$$

which we can rewrite as

$$q_r + \hat{\theta}(w - q_r) = q_p + f(\hat{\theta})(w - q_p). \tag{3}$$

Notice that the solution to equation 2 cannot equal θ^*. Indeed, since $f(\theta^*) = \theta^*$ and $q_r > q_p$, we have

$$\theta^* w + (1 - \theta^*) q_r > f(\theta^*) w + (1 - f(\theta^*)) q_p.$$

Moreover, as equation 3 shows, the left side is an increasing function of θ, and the right side is an decreasing function of θ. So, the solution $\hat{\theta}$ to equation

2 must satisfy $\hat{\theta} < \theta^*$. In summary, the equilibrium defense will be more focused on the run than would be the case if the clock were not an issue.

The offense will use a randomized strategy in which it runs with probability $\hat{p}$ and passes with probability $1-\hat{p}$. We can determine $\hat{p}$ by the condition that $\hat{\theta}$ is optimal against it: The solution to

$$\min_{\theta} \ \hat{p}[\theta w + (1-\theta)q_r] + (1-\hat{p})[f(\theta)w + (1-f(\theta))q_p]$$

must be $\theta = \hat{\theta}$. This implies

$$0 = \hat{p}(w - q_r) + (1-\hat{p})f'(\hat{\theta})(w - q_p),$$

which can be rewritten as

$$\hat{p} = \frac{-f'(\hat{\theta})}{\frac{w-q_r}{w-q_p} - f'(\hat{\theta})}. \tag{4}$$

Since $\hat{\theta} < \theta^*$, the assumption that f is convex implies that

$$-f'(\hat{\theta}) > -f'(\theta^*);$$

and since $q_r > q_p$, we see unambiguously that $\hat{p}$ (given by equation 4) exceeds p^* (given by equation 1). So in equilibrium, the offense will choose to run more often than they would if the clock were not a factor. This is true even though in equilibrium, the defense will be concentrating more on the run.